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Quantum Normal Form and the Harmonic Oscillator with x^6 Perturbation

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The eigenvalue problem of the harmonic oscillator with the x^6 perturbation, $H = \frac{1}{2}(p^2 + x^2 + \lambda x^6)$, is investigated using the method of quantum normal form. The energy eigenvalues are found to be in good agreement with the WKB results.

Recently the energy eigenvalues of an anharmonic oscillator have been derived by Eckardt (1986) (see also Wood and Ali, 1987) employing the so-called Birkhoff–Gustavson normal form approach (Birkhoff, 1927; Gustavson, 1966). In this paper we calculate the energy eigenvalues for the harmonic oscillator with a perturbative term $\frac{1}{2}\lambda x^6$ using the method of the quantum normal forms. The energy eigenvalues of such an oscillator were studied by Banerjee (1978) within the framework of the WKB approximation. One of the aims of the present work will be to compare our results with those of Banerjee's (1978). However, we do not attempt to solve the fundamental problem of quantization of classical Hamiltonian systems.

Consider the ladder operators a^+ and a defined by

$$\begin{aligned} a^+ &= 2^{-1/2}(x - ip) \\ a &= 2^{-1/2}(x + ip) \end{aligned} \tag{1a}$$

satisfying the commutation relations

$$\begin{aligned} [a, a^+] &= 1 \\ [a, a] &= 0 = [a^+, a^+] \end{aligned} \tag{1b}$$

The quantum Hamiltonian of the one-dimensional x^6 oscillator

$$H = \tfrac{1}{2}(p^2 + x^2 + \lambda x^6) \tag{2}$$

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becomes

$$H=(a^{+}a+\tfrac{1}{2})+\tfrac{1}{16}\lambda[(a^{+6}+a^{6})+6(a^{+5}a+a^{+}a^{5})$$
$$+15(a^{+4}a^{2}+a^{+2}a^{4})+60(a^{+3}a+a^{+}a^{3})+15(a^{+4}+a^{4})$$
$$+45(a^{+2}+a^{2})+20a^{+3}a^{3}+90a^{+2}a^{2}+90a^{+}a+15] \tag{3}$$

on using (1). Grouping H as

$$H=H_0+\tfrac{1}{2}\lambda(H_{2\mathrm{N}}+H_{2\mathrm{R}}) \tag{4}$$

where

$$H_0=a^{+}a+\tfrac{1}{2} \tag{5a}$$

$$H_{2\mathrm{N}}=\tfrac{1}{8}(20a^{+3}a^{3}+90a^{+2}a^{2}+90a^{+}a+15) \tag{5b}$$

$$H_{2\mathrm{R}}=\tfrac{1}{8}[(a^{+6}+a^{6})+6(a^{+5}a+a^{+}a^{5})+15(a^{+4}a^{2}+a^{+2}a^{4})$$
$$+60(a^{+3}a+a^{+}a^{3})+15(a^{+4}+a^{4})+45(a^{+2}+a^{2})] \tag{5c}$$

it is immediately obvious that H_0 and $H_{2\mathrm{N}}$ are already in the normal form. This is because both are functions of the number operator $n=a^{+}a$ alone. However, $H_{2\mathrm{R}}$ is not. Note that to $O(\lambda^3)$, H reads (Eckardt, 1986)

$$H=H_0+\lambda\{[S,H_0]+H_{2\mathrm{N}}+H_{2\mathrm{R}}\}+\lambda^2\{[S,H_{2\mathrm{N}}]$$
$$+[S,H_{2\mathrm{R}}]+\tfrac{1}{2}[S,[S,H_0]]\}+O(\lambda^3) \tag{6}$$

To recast H in the normal form, we have to find an operator S such that (Eckardt, 1986; Robnik, 1986)

$$[S,H_0]=-H_{2\mathrm{R}} \tag{7}$$

With H_0 and $H_{2\mathrm{R}}$ given by equations (5a) and (5c), respectively, S can be readily found:

$$S=(\lambda/96)[2(a^{+6}-a^{6})+18(a^{+5}a-a^{+}a^{5})$$
$$+90(a^{+4}a^{2}-a^{+2}a^{4})+360(a^{+3}a-a^{+}a^{3})$$
$$+45(a^{+4}-a^{4})+540(a^{+2}-a^{2})] \tag{8}$$

The crucial point is that if our S is known, then it is a simple exercise to find H. Corresponding to equation (3), we obtain an expression for the eigenvalues from equation (6):

$$E_n=n+\tfrac{1}{2}+(5\lambda/8)(2n^3+3n^2+4n+3/2)$$
$$+(\lambda^2/256)(1167n^5+3255n^4+10{,}861n^3+13{,}077n^2$$
$$+10{,}567n+3495)+O(\lambda^3) \tag{9}$$

Table I. Energy Eigenvalues[a] for the Anharmonic Oscillator for Various Values of the Parameter λ

Parameter value	Ground state		First excited state		Second excited state	
	Present	Banerjee	Present	Banerjee	Present	Banerjee
0.00001	1.000	1.000	3.000	3.000	5.001	5.000
0.0001	1.000	1.000	3.001	3.001	5.005	5.005
0.001	1.002	1.001	3.013	3.012	5.049	5.045
0.01	1.021	1.017	3.164	3.107	5.605	5.347

[a] The energy eigenvalues in the present calculation have been multiplied by a factor of 2 in order to compare them with the corresponding results of Banerjee (1978).

Now we are in a position to compare the energy eigenvalues of the harmonic oscillator with a perturbation term with the WKB results of Banerjee (1978). Table I gives the first three eigenvalues and quotes the results of Banerjee for comparison.

To conclude, the transformation to the normal form via a series of unitary transformations can be carried out to any desired order in λ. The series is exactly identical to the Rayleigh-Schrödinger perturbation series, and hence is divergent for all $\lambda > 0$ (Banerjee, 1978), the more so for large n. However, the agreement is quite impressive for small λ and n, thus lending credence to the view (Ali, 1985) that the Birkhoff-Gustavson normal form yields fairly reasonable results which may be difficult to obtain otherwise.

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REFERENCES

Ali, M. K. (1985). *Journal of Mathematical Physics*, **26**, 2565.

Banerjee, K. (1978). *Proceedings of the Royal Society of London*, **364**, 265.

Birkhoff, G. D. (1927). *Dynamical Systems*, AMS Colloquium Publications, New York, Volume IX.

Eckardt, B. (1986). *Journal of Physics A*, **19**, 2961.

Gustavson, F. G. (1966). *Astronomical Journal*, **71**, 547.

Robnik, M. (1986). *Journal of Physics A*, **19**, L841.

Wood, W. R. and Ali, M. K. (1987). *Journal of Physics A*, **20**, 351.